



Politechnika
Wrocławska

Metody numeryczne w fizyce

W11OPA-SM0060G

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Wykład 9

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HR EXCELLENCE IN RESEARCH



Plan

- Zagadnienie początkowe
- Metoda Eulera
- Metody Rungego-Kutty
- Kontrola wielkość błędu

Zagadnienie początkowe

Typowe zagadnienie początkowe opisane jest równaniem

$$\frac{df}{dt} = g(t, f), \quad f(t_0) = f_0.$$

W zagadnieniu początkowym może występować więcej zmiennych

$$\frac{d\mathbf{f}}{dt} = \mathbf{g}(t, \mathbf{f}), \quad \mathbf{f}(t_0) = \mathbf{f}_0.$$

Metoda Eulera

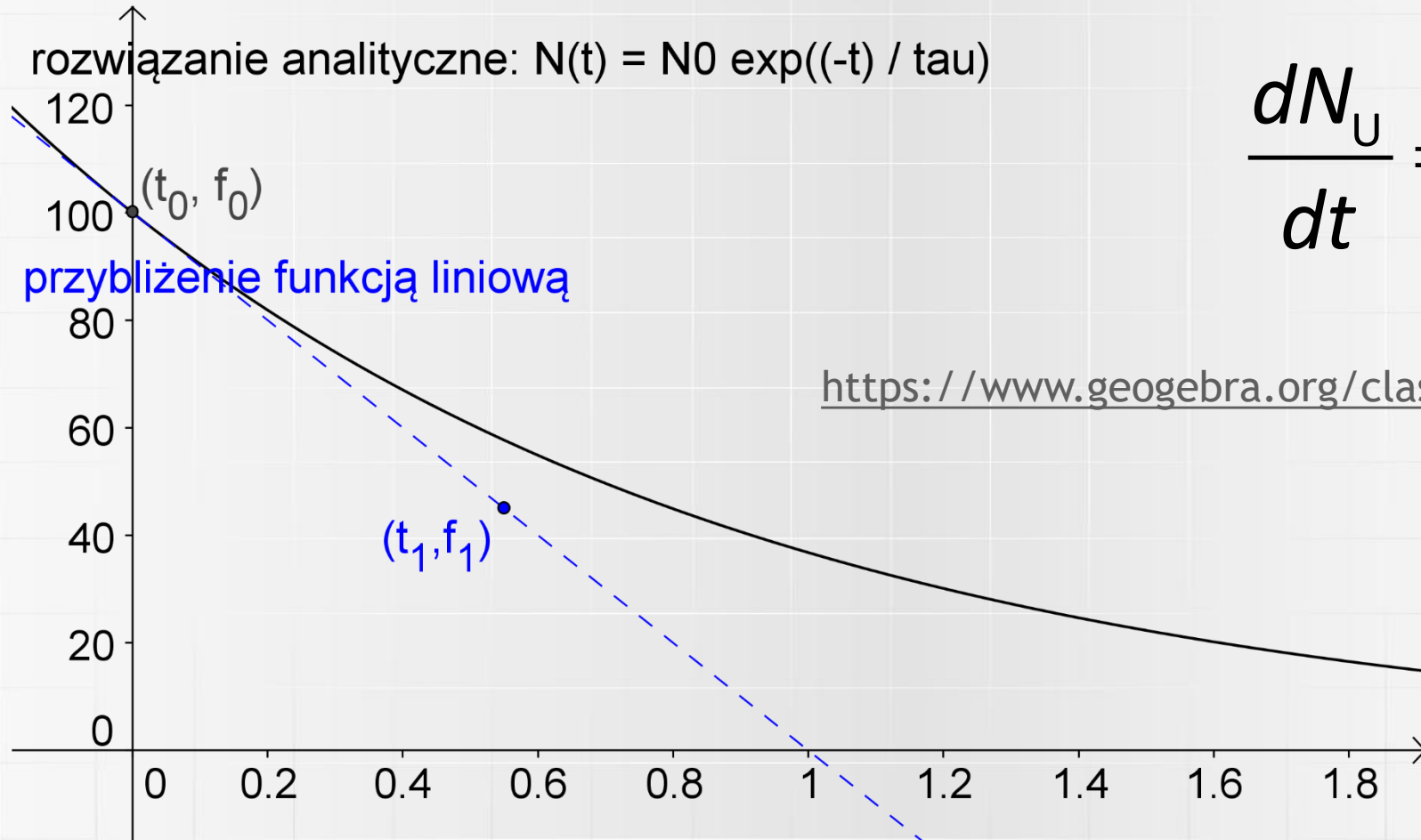
$$\frac{df}{dt} = g(t, f), \quad f(t_0) = f_0.$$

$$\frac{f_{n+1} - f_n}{t_{n+1} - t_n} \approx g(t_n, f_n) = g_n$$

$$\frac{f_{n+1} - f_n}{h} \approx g_n$$

$$f_{n+1} = f_n + hg_n + O(h^2)$$

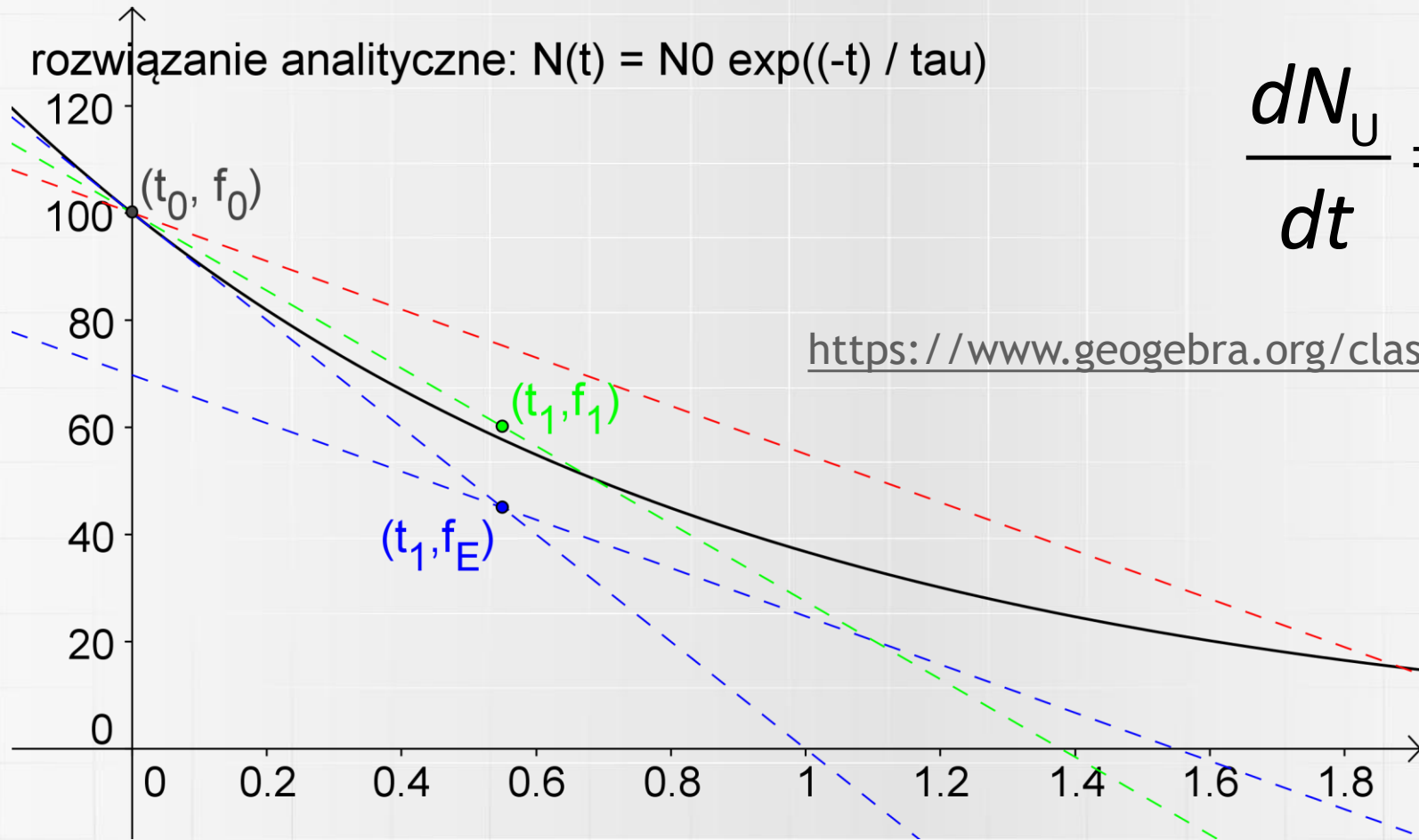
Metoda Eulera



$$\frac{dN_U}{dt} = -\frac{N_U}{\tau}$$

<https://www.geogebra.org/classic/sb2fjudh>

Metoda Heuna



$$\frac{dN_U}{dt} = -\frac{N_U}{\tau}$$

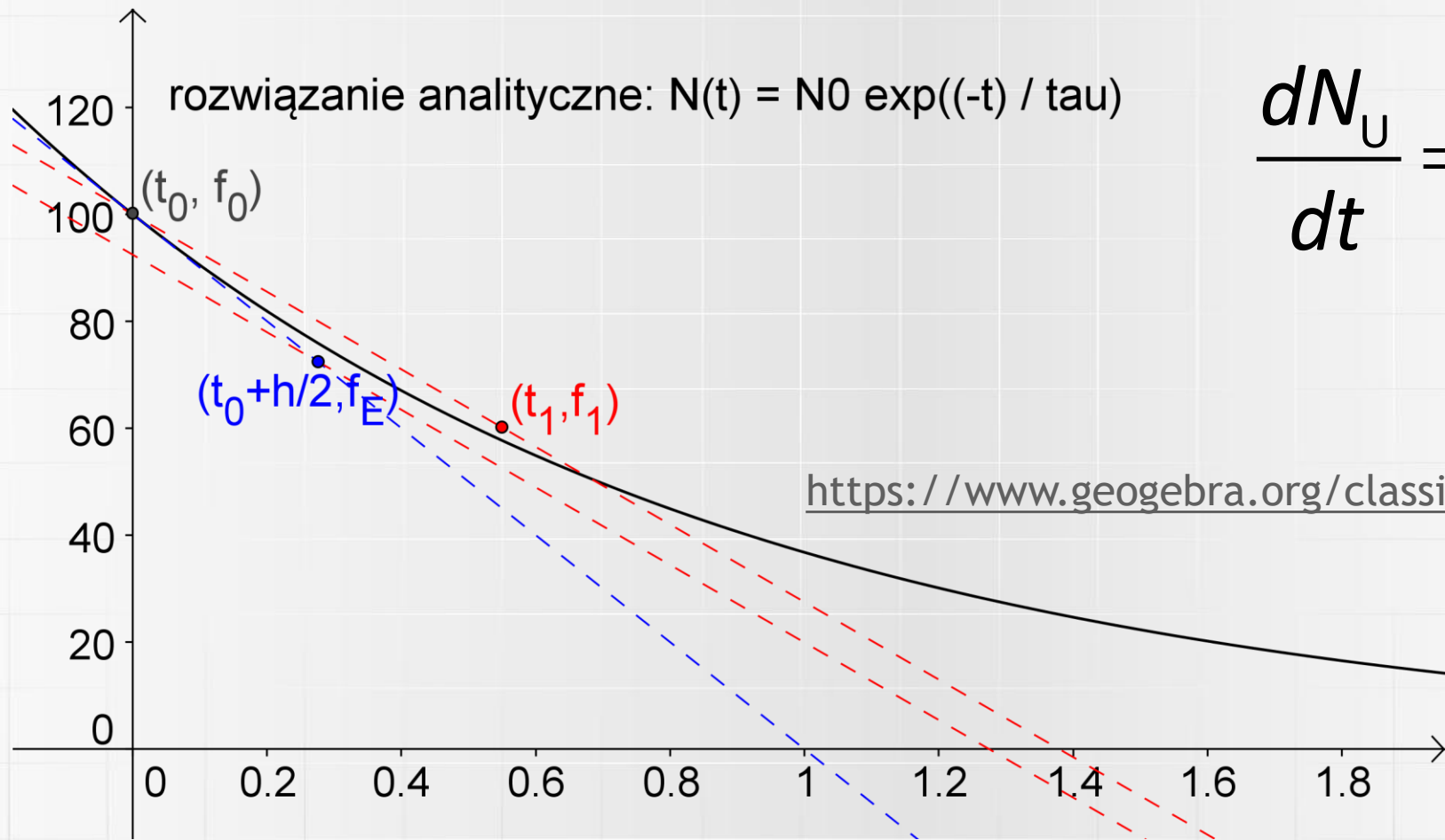
<https://www.geogebra.org/classic/spkxdnwf>

Metoda Heuna

$$f(t+h) \approx f(t) + \frac{1}{2}(k_1 + k_2)$$

$$\begin{aligned} k_1 &= hg(t, f) \\ k_2 &= hg(t+h, f+k_1) \end{aligned}$$

Metoda punktu pośredniego



$$\frac{dN_U}{dt} = -\frac{N_U}{\tau}$$

Metoda punktu pośredniego

$$f(t+h) \approx f(t) + k_2$$

$$k_1 = hg(t, f)$$

$$k_2 = hg\left(t + \frac{h}{2}, f + \frac{k_1}{2}\right)$$

Metody Rungego-Kutty rzędu drugiego

$$f(t+h) \approx f(t) + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t+h, f(t) + k_1)$$

$$f(t+h) \approx f(t) + k_2$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg\left(t + \frac{h}{2}, f(t) + \frac{k_1}{2}\right)$$

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t + v_{21}h, f(t) + v_{21}k_1)$$

Metody Rungego-Kutty rzędu drugiego

$$\frac{df}{dt} = g(t, f) \quad f(t+h) = f(t) + h \frac{df}{dt} \Big|_t + \frac{h^2}{2} \frac{d^2 f}{dt^2} \Big|_t + O(h^3)$$

$$f(t+h) = f(t) + hg(t, f(t)) + \frac{h^2}{2} \frac{d}{dt} g \Big|_t + O(h^3)$$

$$f(t+h) = f(t) + hg(t, f(t)) + \frac{h^2}{2} \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f} \frac{df}{dt} \right) \Big|_t + O(h^3)$$

$$f(t+h) = f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

Metody Rungego-Kutty rzędu drugiego

$$k_1 = hg(t, f(t))$$

$$k_2 = h \left[g(t + \nu_{21}h, f(t) + \nu_{21}k_1) \right]$$

$$k_2 = h \left[g(t + \nu_{21}h, f(t) + \nu_{21}hg(t, f(t))) \right]$$

$$k_2 = h \left[g(t, f(t)) + \frac{\partial g}{\partial t} \Big|_t \nu_{21}h + \frac{\partial g}{\partial f} \Big|_t \nu_{21}hg(t, f(t)) + O(h^2) \right]$$

$$k_2 = hg(t, f(t)) + \frac{\partial g}{\partial t} \Big|_t \nu_{21}h^2 + \frac{\partial g}{\partial f} \Big|_t \nu_{21}h^2 g(t, f(t)) + O(h^3)$$

Metody Rungego-Kutty rzędu drugiego

$$k_2 = hg(t, f(t)) + \frac{\partial g}{\partial t} \Big|_t v_{21} h^2 + \frac{\partial g}{\partial f} \Big|_t v_{21} h^2 g(t, f(t)) + O(h^3)$$

$$k_2 = hg(t, f(t)) + v_{21} h^2 \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f} g \right) \Big|_t + O(h^3)$$

$$f(t+h) = f(t) + \alpha_1 k_1 + \alpha_2 k_2 \qquad k_1 = hg(t, f(t))$$

$$f(t+h) = f(t) + \alpha_1 hg(t, f(t)) + \alpha_2 hg(t, f(t)) + \\ + \alpha_2 v_{21} h^2 \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f} g \right) \Big|_t + O(h^3)$$

Metody Rungego-Kutty rzędu drugiego

$$f(t+h) = f(t) + (\alpha_1 + \alpha_2)hg(t, f(t)) + \\ + \alpha_2\nu_{21}h^2 \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f}g \right) \Big|_t + O(h^3)$$

$$f(t+h) = f(t) + hg(t, f(t)) + \frac{h^2}{2} \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f} \frac{df}{dt} \right) \Big|_t + O(h^3)$$

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_2\nu_{21} = \frac{1}{2} \end{cases}$$

$$\begin{cases} \alpha_1 = \frac{1}{2} \\ \alpha_2 = \frac{1}{2} \\ \nu_{21} = 1 \end{cases}$$

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 1 \\ \nu_{21} = \frac{1}{2} \end{cases}$$

Metody Rungego-Kutty rzędu drugiego

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t + \nu_{21}h, f(t) + \nu_{21}k_1)$$

ν_{21}	
α_1	α_2

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_2 \nu_{21} = \frac{1}{2} \end{cases}$$

1	
$\frac{1}{2}$	$\frac{1}{2}$

$$\begin{cases} \alpha_1 = \frac{1}{2} \\ \alpha_2 = \frac{1}{2} \\ \nu_{21} = 1 \end{cases}$$

$\frac{1}{2}$	
0	1

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 1 \\ \nu_{21} = \frac{1}{2} \end{cases}$$

Metody Rungego-Kutty

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_n k_n$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t + v_{21}h, f(t) + v_{21}k_1)$$

$$k_3 = hg(t + v_{31}h + v_{32}h, f(t) + v_{31}k_1 + v_{32}k_2)$$

$$\vdots$$

$$k_n = hg\left(t + h \sum_{i=1}^{n-1} v_{ni}, f(t) + \sum_{i=1}^{n-1} v_{ni} k_i\right)$$

Metoda Rungego-Kutty rzędu czwartego

$$f(t+h) \approx f(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg\left(t + \frac{1}{2}h, f(t) + \frac{1}{2}k_1\right)$$

$$k_3 = hg\left(t + \frac{1}{2}h, f(t) + \frac{1}{2}k_2\right)$$

$$k_4 = hg(t+h, f(t) + k_3)$$

1/2			
0	1/2		
0	0	1	
1/6	1/3	1/3	1/6

Metoda Rungego-Kutty rzędu czwartego

$$f(t+h) = \tilde{f}(t+h) + Ch^5$$

$$f(t+h) = \hat{f}(t+h) + 2C\left(\frac{h}{2}\right)^5$$

$$0 = \tilde{f}(t+h) - \hat{f}(t+h) + \frac{15}{16}Ch^5$$

$$Ch^5 = \frac{16}{15}[\hat{f}(t+h) - \tilde{f}(t+h)] \approx \hat{f}(t+h) - \tilde{f}(t+h)$$

Metody adaptacyjne Rungego-Kutty

Metoda Rungego-Kutty-Fehlberga

Liczba obliczonych wartości funkcji	1	2	3	4	5	6	7	8
Maksymalny rząd metody	1	2	3	4	4	5	6	6

$$\hat{f}(t+h) := f(t) + \sum_{i=1}^6 \alpha_i k_i$$

$$\tilde{f}(t+h) := f(t) + \sum_{i=1}^6 \beta_i k_i$$

$$k_i := hg \left(t + h \sum_{j=1}^{i-1} \nu_{ij}, f(t) + \sum_{j=1}^{i-1} \nu_{ij} k_j \right)$$

Metody adaptacyjne Rungego-Kutty

Metoda Rungego-Kutty-Fehlberga

i	α_i	$\alpha_i - \beta_i$	d_{i1}	d_{i2}	d_{i3}	d_{i4}	d_{i5}
1	$\frac{16}{135}$	$\frac{1}{360}$					
2	0	0	$\frac{1}{4}$				
3	$\frac{6656}{12825}$	$-\frac{128}{4275}$	$\frac{3}{32}$	$\frac{9}{32}$			
4	$\frac{28561}{56430}$	$-\frac{2197}{75240}$	$\frac{1932}{21967}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$		
5	$-\frac{9}{50}$	$\frac{1}{50}$	$\frac{439}{216}$	-8	$\frac{3680}{513}$	$-\frac{845}{4104}$	
6	$\frac{2}{55}$	$\frac{2}{55}$	$-\frac{8}{27}$	2	$-\frac{3544}{2565}$	$\frac{1859}{4104}$	$-\frac{11}{40}$

Metody adaptacyjne Rungego-Kutty

Metoda Rungego-Kutty-Fehlberga

$$e := \hat{f}(t+h) - \tilde{f}(t+h) = \sum_{i=1}^6 (\alpha_i - \beta_i) k_i$$

$$|e| < \delta$$

$$e \approx Ch^5$$

$$|e| < \delta / 128$$

Metody adaptacyjne Rungego-Kutty

ode23 Bogacki-Shampine

i	α_i	β_i	d_{i1}	d_{i2}	d_{i3}
1	$\frac{2}{9}$	$\frac{7}{24}$			
2	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$		
3	$\frac{4}{9}$	$\frac{1}{3}$	0	$\frac{3}{4}$	
4	0	$\frac{1}{8}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

$$\hat{f}(t+h) := f(t) + \sum_{i=1}^4 \alpha_i k_i$$

$$\tilde{f}(t+h) := f(t) + \sum_{i=1}^4 \beta_i k_i$$

$$k_i := hg \left(t + h \sum_{j=1}^{i-1} d_{ij}, f(t) + \sum_{j=1}^{i-1} d_{ij} k_j \right)$$

Podsumowanie (1)

- Zagadnienie początkowe

$$\frac{df}{dt} = g(t, f), \quad f(t_0) = f_0$$

- Metoda Eulera

$$f_{n+1} = f_n + hg_n + O(h^2)$$

- Metody Rungego-Kutty

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t + \nu_{21}h, f(t) + \nu_{21}k_1)$$

Podsumowanie (2)

- Metody Rungego-Kutty

$$f(t + h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_n k_n$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t + v_{21}h, f(t) + v_{21}k_1)$$

$$k_3 = hg(t + v_{31}h + v_{32}h, f(t) + v_{31}k_1 + v_{32}k_2)$$

⋮

$$k_n = hg\left(t + h \sum_{i=1}^{n-1} v_{ni}, f(t) + \sum_{i=1}^{n-1} v_{ni} k_i\right)$$

- Kontrola wielkość błędu