Politechnika Wrocławska

Metody numeryczne w fizyce

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Wykład 7

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Plan wykładu

- Zagadnienie początkowe
- Metoda Eulera
- Metody Rungego-Kutty
  - metoda Heuna
  - metoda punktu pośredniego
  - metoda rzędu czwartego
- Kontrola wielkość błędu

Zagadnienie początkowe

Typowe zagadnienie początkowe opisane jest równaniem

$$\frac{df}{dt} = g(t, f), \quad f(t_0) = f_0.$$

W zagadnieniu początkowym może występować więcej zmiennych

$$\frac{d\mathbf{f}}{dt} = \mathbf{g}(t, \mathbf{f}), \quad \mathbf{f}(t_0) = \mathbf{f}_0.$$

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Metoda Eulera

$$\frac{df}{dt} = g(t,f), \quad f(t_0) = f_0.$$

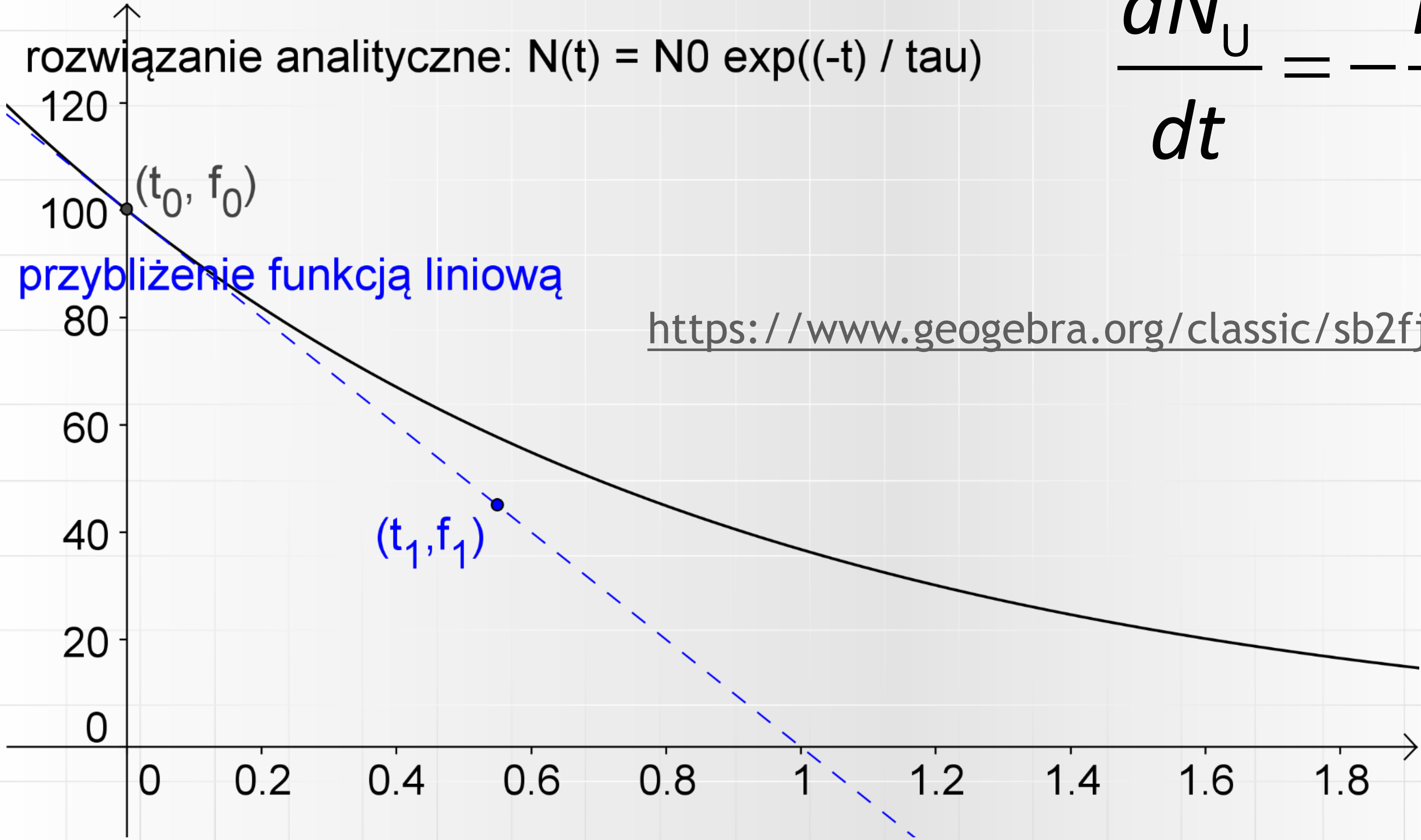
$$\frac{f_{n+1} - f_n}{t_{n+1} - t_n} \approx g(t_n, f_n) = g_n$$

$$\frac{f_{n+1} - f_n}{h} \approx g_n$$

$$\boxed{f_{n+1} = f_n + hg_n + O(h^2)}$$

Metoda Eulera

$$\frac{dN_U}{dt} = -\frac{N_U}{\tau}$$

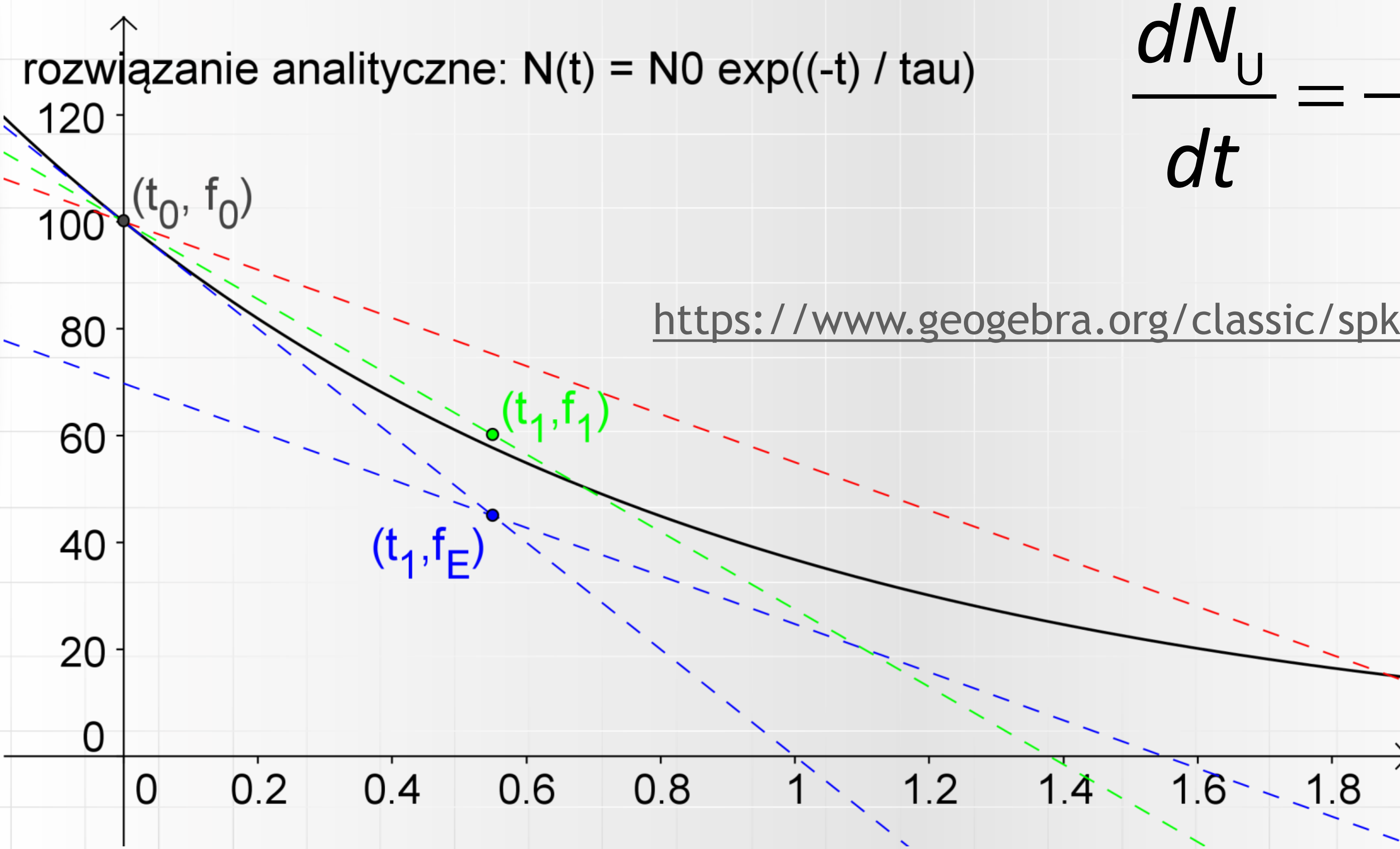


https://www.geogebra.org/classic/sb2fjudh

Metoda Heuna

$$\frac{dN_U}{dt} = -\frac{N_U}{\tau}$$

https://www.geogebra.org/classic/spkxdnwf

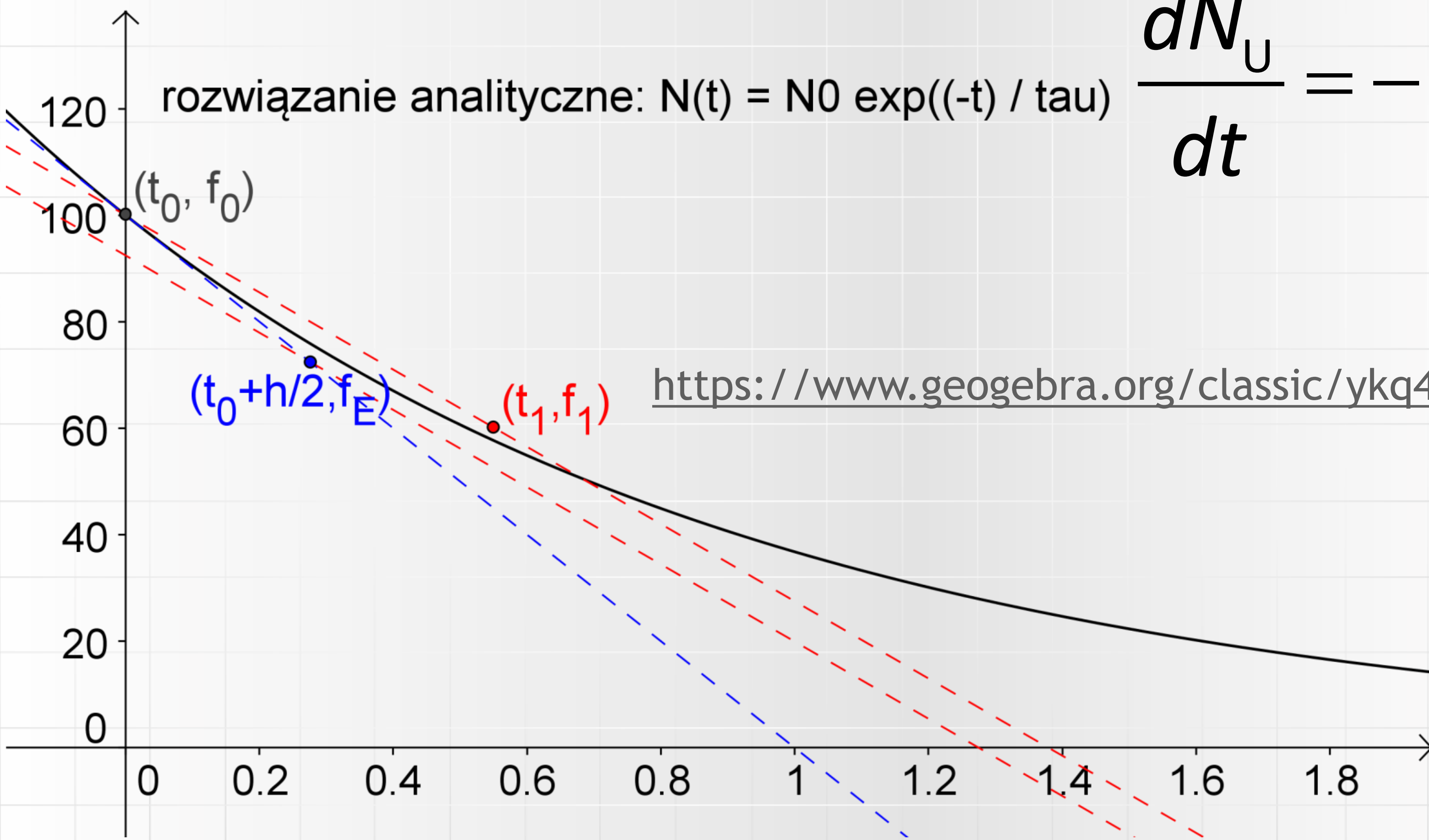


Metoda Heuna

$$f(t+h) \approx f(t) + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hg(t, f)$$
$$k_2 = hg(t+h, f+k_1)$$

Metoda punktu pośredniego



$$\frac{dN_U}{dt} = -\frac{N_U}{\tau}$$

https://www.geogebra.org/classic/ykq4ynag

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Metoda punktu pośredniego

$$f(t+h) \approx f(t) + k_2$$

$$k_1 = hg(t, f)$$

$$k_2 = hg\left(t + \frac{h}{2}, f + \frac{k_1}{2}\right)$$

Metody Rungego-Kutty rzędu drugiego

$$f(t+h) \approx f(t) + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hg(t, f(t))$$
$$k_2 = hg(t+h, f(t) + k_1)$$

$$f(t+h) \approx f(t) + k_2$$

$$k_1 = hg(t, f(t))$$
$$k_2 = hg\left(t + \frac{h}{2}, f(t) + \frac{k_1}{2}\right)$$

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

$$k_1 = hg(t, f(t))$$
$$k_2 = hg(t + \nu_{21} h, f(t) + \nu_{21} k_1)$$

Metody Rungego-Kutty rzędu drugiego

$$\frac{df}{dt} = g(t,f) \qquad f(t+h) = f(t) + h\left.\frac{df}{dt}\right|_t + \frac{h^2}{2}\left.\frac{d^2f}{dt^2}\right|_t + O(h^3)$$

$$f(t+h) = f(t) + hg(t,f(t)) + \frac{h^2}{2}\frac{d}{dt}g\Big|_t + O(h^3)$$

$$f(t+h) = f(t) + hg(t,f(t)) + \frac{h^2}{2}\left.\left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f}\frac{df}{dt}\right)\right|_t + O(h^3)$$

$$f(t+h) = f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

Metody Rungego-Kutty rzędu drugiego

$$k_1 = hg\left(t, f\left(t\right)\right)$$

$$k_2 = h\left[g\left(t + \nu_{21}h, f\left(t\right) + \nu_{21}k_1\right)\right]$$

$$k_2 = h\left[g\left(t + \nu_{21}h, f\left(t\right) + \nu_{21}hg\left(t, f\left(t\right)\right)\right)\right]$$

$$k_2 = h\left[g\left(t, f\left(t\right)\right) + \left.\frac{\partial g}{\partial t}\right|_t \nu_{21}h + \left.\frac{\partial g}{\partial f}\right|_t \nu_{21}hg\left(t, f\left(t\right)\right) + O\left(h^2\right)\right]$$

$$k_2 = hg\left(t, f\left(t\right)\right) + \left.\frac{\partial g}{\partial t}\right|_t \nu_{21}h^2 + \left.\frac{\partial g}{\partial f}\right|_t \nu_{21}h^2 g\left(t, f\left(t\right)\right) + O\left(h^3\right)$$

Metody Rungego-Kutty rzędu drugiego

$$k_2 = hg(t, f(t)) + \left.\frac{\partial g}{\partial t}\right|_t \nu_{21} h^2 + \left.\frac{\partial g}{\partial f}\right|_t \nu_{21} h^2 g(t, f(t)) + O(h^3)$$

$$k_2 = hg(t, f(t)) + \nu_{21} h^2 \left.\left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f} g \right)\right|_t + O(h^3)$$

$$f(t+h) = f(t) + \alpha_1 k_1 + \alpha_2 k_2 \qquad k_1 = hg(t, f(t))$$

$$f(t+h) = f(t) + \alpha_1 hg(t, f(t)) + \alpha_2 hg(t, f(t)) +$$

$$+ \alpha_2 \nu_{21} h^2 \left.\left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial f} g \right)\right|_t + O(h^3)$$

Metody Rungego-Kutty rzędu drugiego

$$f\left(t+h\right)=f\left(t\right)+\left(\alpha_1+\alpha_2\right)hg\left(t,f\left(t\right)\right)+$$

$$+\alpha_2 v_{21}h^2\left(\frac{\partial g}{\partial t}+\frac{\partial g}{\partial f}g\right)\Bigg|_t+O\left(h^3\right)$$

$$f\left(t+h\right)=f\left(t\right)+hg\left(t,f\left(t\right)\right)+\frac{h^2}{2}\left(\frac{\partial g}{\partial t}+\frac{\partial g}{\partial f}\frac{df}{dt}\right)\Bigg|_t+O\left(h^3\right)$$

$$\begin{cases}\alpha_1+\alpha_2=1\\ \alpha_2 v_{21}=\frac{1}{2}\end{cases}\qquad \begin{cases}\alpha_1=\frac{1}{2}\\ \alpha_2=\frac{1}{2}\\ v_{21}=1\end{cases}\qquad \begin{cases}\alpha_1=0\\ \alpha_2=1\\ v_{21}=\frac{1}{2}\end{cases}$$

Metody Rungego-Kutty rzędu drugiego

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

$$k_1 = hg(t, f(t))$$
$$k_2 = hg(t + \nu_{21}h, f(t) + \nu_{21}k_1)$$

ν_{21}	
α_1	α_2

1	
½	½

½	
0	1

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_2 \nu_{21} = \dfrac{1}{2} \end{cases}$$

$$\begin{cases} \alpha_1 = \frac{1}{2} \\ \alpha_2 = \frac{1}{2} \\ \nu_{21} = 1 \end{cases}$$

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 1 \\ \nu_{21} = \frac{1}{2} \end{cases}$$

Metody Rungego-Kutty

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2 + \ldots + \alpha_n k_n$$

$$\begin{aligned}
k_1 &= hg\left(t, f(t)\right) \\
k_2 &= hg\left(t + \nu_{21}h, f(t) + \nu_{21}k_1\right) \\
k_3 &= hg\left(t + \nu_{31}h + \nu_{32}h, f(t) + \nu_{31}k_1 + \nu_{32}k_2\right) \\
&\vdots \\
k_n &= hg\left(t + h\sum_{i=1}^{n-1}\nu_{ni}, f(t) + \sum_{i=1}^{n-1}\nu_{ni}k_i\right)
\end{aligned}$$

Metoda Rungego-Kutty rzędu czwartego

$$f(t+h) \approx f(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg\left(t + \frac{1}{2}h, f(t) + \frac{1}{2}k_1\right)$$

$$k_3 = hg\left(t + \frac{1}{2}h, f(t) + \frac{1}{2}k_2\right)$$

$$k_4 = hg(t+h, f(t) + k_3)$$

½			
0	½		
0	0	1	
1/6	1/3	1/3	1/6

Metoda Rungego-Kutty rzędu czwartego

$$f(t+h) = \tilde{f}(t+h) + Ch^5$$

$$f(t+h) = \hat{f}(t+h) + 2C\left(h/2\right)^5$$

$$0 = \tilde{f}(t+h) - \hat{f}(t+h) + \frac{15}{16}Ch^5$$

$$Ch^5 = \frac{16}{15}\left[\hat{f}(t+h) - \tilde{f}(t+h)\right] \approx \hat{f}(t+h) - \tilde{f}(t+h)$$

Metody adaptacyjne Rungego-Kutty
Metoda Rungego-Kutty-Fehlberga

Liczba obliczonych wartości funkcji	1	2	3	4	5	6	7	8
Maksymalny rząd metody	1	2	3	4	4	5	6	6

$$\hat{f}(t+h) := f(t) + \sum_{i=1}^{6} \alpha_i k_i$$

$$\tilde{f}(t+h) := f(t) + \sum_{i=1}^{6} \beta_i k_i$$

$$k_i := hg\left(t + h\sum_{j=1}^{i-1} \nu_{ij}, f(t) + \sum_{j=1}^{i-1} \nu_{ij} k_j \right)$$

Metody adaptacyjne Rungego-Kutty
Metoda Rungego-Kutty-Fehlberga

i	α_i	α_i-β_i	d_{i1}	d_{i2}	d_{i3}	d_{i4}	d_{i5}
1	$\frac{16}{135}$	$\frac{1}{360}$					
2	0	0	$\frac{1}{4}$				
3	$\frac{6656}{12825}$	$-\frac{128}{4275}$	$\frac{3}{32}$	$\frac{9}{32}$			
4	$\frac{28561}{56430}$	$-\frac{2197}{75240}$	$\frac{1932}{21967}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$		
5	$-\frac{9}{50}$	$\frac{1}{50}$	$\frac{439}{216}$	-8	$\frac{3680}{513}$	$-\frac{845}{4104}$	
6	$\frac{2}{55}$	$\frac{2}{55}$	$-\frac{8}{27}$	2	$-\frac{3544}{2565}$	$\frac{1859}{4104}$	$-\frac{11}{40}$

Metody adaptacyjne Rungego-Kutty

Metoda Rungego-Kutty-Fehlberga

$$e := \hat{f}(t+h) - \tilde{f}(t+h) = \sum_{i=1}^{6} (\alpha_i - \beta_i) k_i$$

$$|e| < \delta$$

$$e \approx Ch^5$$

$$|e| < \delta / 128$$

Metody adaptacyjne Rungego-Kutty
ode23 Bogacki-Shampine

i	α_i	β_i	d_{i1}	d_{i2}	d_{i3}
1	$\frac{2}{9}$	$\frac{7}{24}$			
2	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$		
3	$\frac{4}{9}$	$\frac{1}{3}$	0	$\frac{3}{4}$	
4	0	$\frac{1}{8}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

$$\hat{f}(t+h) := f(t) + \sum_{i=1}^{4} \alpha_i k_i$$

$$\tilde{f}(t+h) := f(t) + \sum_{i=1}^{4} \beta_i k_i$$

$$k_i := hg\left(t + h\sum_{j=1}^{i-1} d_{ij}, f(t) + \sum_{j=1}^{i-1} d_{ij} k_j\right)$$

Podsumowanie (1)

- Zagadnienie początkowe $\frac{d\mathbf{f}}{dt} = \mathbf{g}(t, \mathbf{f}), \quad \mathbf{f}(t_0) = \mathbf{f}_0$

- Metoda Eulera $f_{n+1} = f_n + hg_n + O(h^2)$

- Metody Rungego-Kutty

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2$$

$$k_1 = hg(t, f(t))$$

$$k_2 = hg(t + \nu_{21} h, f(t) + \nu_{21} k_1)$$

Podsumowanie (2)

- Metody Rungego-Kutty

$$f(t+h) \approx f(t) + \alpha_1 k_1 + \alpha_2 k_2 + \ldots + \alpha_n k_n$$

$$\begin{aligned}
k_1 &= hg(t, f(t)) \\
k_2 &= hg(t + \nu_{21}h, f(t) + \nu_{21}k_1) \\
k_3 &= hg(t + \nu_{31}h + \nu_{32}h, f(t) + \nu_{31}k_1 + \nu_{32}k_2) \\
&\vdots \\
k_n &= hg\left(t + h\sum_{i=1}^{n-1}\nu_{ni}, f(t) + \sum_{i=1}^{n-1}\nu_{ni}k_i\right)
\end{aligned}$$

- Kontrola wielkość błędu